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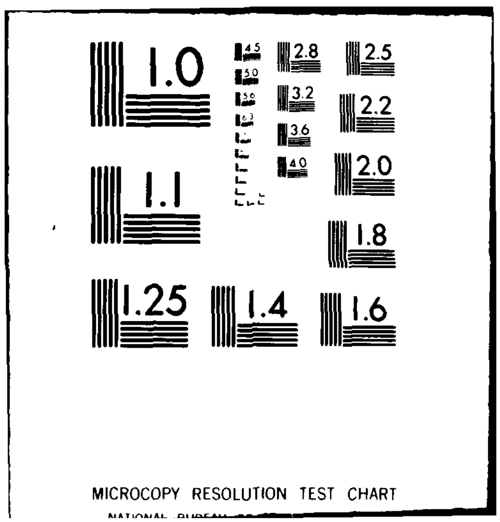
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THE FORM OF THE SOLUTIONS OF THE LINEAR INTEGRO-DIFFERENTIAL EQUATIONS OF SUBSONIC AEROELASTICITY

by

D. L. Woodcock

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(6) THE FORM OF THE SOLUTIONS OF THE LINEAR INTEGRO-DIFFERENTIAL EQUATIONS OF SUBSONIC AEROELASTICITY

by

(10) D. L. Woodcock

SUMMARY

The solution of the subsonic flutter problem, when the commonly used linear differential equation model is replaced by the more correct linear integro-differential equation model, is studied and the nature of the system's free motion established. The different forms appropriate to two-dimensional and three-dimensional flow, and to the cases when the system has a zero characteristic value, are developed in detail. It is shown that, for large time t , the behaviour can variously be like ~~1, $\log^{-1} t$, t^{-2} or t^{-3}~~.

1, 1/log, 1/t sq, 1/t cubed.

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THE FORM OF THE SOLUTIONS OF THE LINEAR INTEGRO-DIFFERENTIAL EQUATIONS OF SUBSONIC AEROELASTICITY

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CORRIGENDA

1 The analysis of this paper has been based on the assumption that the Laplace transform $\bar{G}\left(\frac{p}{v}\right)$ of that part, $G(v\tau)$, of the indicial aerodynamic matrix (equation (2)) which remains when the constant and impulsive terms are excluded, had the form given by Milne[2] based on the work of Garner and Milne[10]. This form, reproduced in equation (11) is

$$\bar{G}\left(\frac{p}{v}\right) = \sum_{s=0}^{\infty} \left(\frac{p}{v}\right)^{s} L_{s} + \log\left(\frac{p}{v}\right) \sum_{s=0}^{\infty} \left(\frac{p}{v}\right)^{s} N_{s} \qquad (11)$$

where N_0 is zero except in the two-dimensional case. However a closer examination of Refs 2 and 10 reveals that their authors only showed the existence in the three-dimensional case of the constant and $\left(\frac{p}{v}\right) \log \left(\frac{p}{v}\right)$ terms in (11). Moreover, using the series expansion[17] for the three-dimensional kernel function*, one finds that $\bar{G}$ has the form

* The kernel is a power series in p, plus $p^2 \log p$ times another power series in p.

$$\bar{G}\left(\frac{p}{v}\right) = \sum_{s=0}^{\infty} \left(\frac{p}{v}\right)^{s} L_s + \sum_{r=1}^{\infty} \sum_{s=r}^{\infty} \log^r\left(\frac{p}{v}\right)\left(\frac{p}{v}\right)^{r+s-1} N_{rs} \quad . \qquad \text{(C-1)}$$

Further confirmation of the existence of the higher powers of $\log\left(\frac{p}{v}\right)$ is obtained when we expand the known form (equations (12) and (13)) of $\bar{G}\left(\frac{p}{v}\right)$ for the two-dimensional incompressible case. The series then has the form

$$\bar{G}\left(\frac{p}{v}\right) = \sum_{s=0}^{\infty} \left(\frac{p}{s}\right)^{s} L_s + \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} \log^r\left(\frac{p}{v}\right)\left(\frac{p}{v}\right)^{r+s-1} N_{rs} \quad . \qquad \text{(C-2)}$$

The same form of series is also obtained when one uses the series expansion[18] of the kernel function for two-dimensional compressible flow.

Taking (C-2) as the general form instead of (11), and noting that N_{rs} is zero in the three-dimensional case when $r > s$, the asymptotic expansion of $G(v\tau)$ is changed from (3) to

$$G(v\tau) \sim \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} G_{rs} \frac{\log^r(v\tau)}{(v\tau)^{r+s+1}} \qquad \text{(C-3)}$$

where, in the three-dimensional case G_{rs} is zero for $r \gg s$. The coefficients in (C-3) are not as simply related to those of (C-2) as were those of (3) to those of (11). Any particular G_{rs} is a linear combination of the $N_{r+u+1,s-u}$ $(u = 0 \rightarrow s)$. For example we have

$$\left.\begin{aligned} G_{00} &= - N_{10} \\ G_{01} &= N_{11} + (2 - 2\gamma)N_{20} \\ G_{10} &= - 2N_{20} \\ G_{11} &= 4N_{21} + (18 - 12\gamma)N_{30} \\ &\text{etc.} \end{aligned}\right\} \qquad \text{(C-4)}$$

This change - (C-2) instead of (11) - makes no change to the character of the subsequent analysis; and, in particular, no change in every case to the form of the dominant term, for large time, of the fundamental solution X_0 . There is

some apparent modification of the overall form of X_0 in the three-dimensional case. We find instead of (46)

$$X_0 = v \sum_{i=1}^{k} W_i e^{\lambda_i \tau} + v^3 \sum_{s=3}^{\infty} \sum_{r=0}^{\infty} \log^r (v\tau) \frac{S_{rs}(v)}{(v\tau)^{2r+s}} . \qquad \text{(C-5)}$$

The change of form is only apparent, for the fact was overlooked that the S_{rs} of equation (46) were zero for $s < (r + 3)$; and so putting $s = (u + r)$ in (46)* transforms it into a form like (C-5).

When the characteristic equation has a zero root the form is still that given in equation (47) though the coefficients U_{rs} , similar to the S_{rs} appearing in (C-5), are now derived differently. In the two-dimensional case the fundamental solutions are those given in equations (39) and (51) with a similar proviso about the coefficients.

2 Additional references

No.	Author	Title, etc
17	C.E. Watkins H.L. Runyan D.S. Woolston	On the kernel function of the integral equation relating the lift and downwash distributions of oscillating finite wings in subsonic flow. NACA Report 1234 (1955)
18	Deborah J. Salmond	Evaluation of two-dimensional subsonic oscillatory airforce coefficients and loading distributions. RAE Technical Report 79096 (1979)

3 Corrections to the paper as written, *ie* without making the change suggested in section 1 above.

p5 equation (10) - replace $v \to \infty$ by $\nu \to \infty$

p7 equation (16) - replace ν^2 by v^2

p8 line before equation (22) - insert at end: , with the wing chord as the reference length

line after equation (22) - add asterisk to choice and insert footnote:
* For finite aspect ratio wings appropriate values of p_0 will be larger. The results of Ref 9 suggest for example a value roughly twice as big for an aspect ratio 4 wing.

p12 first line - replace 'T_{0s} $(s \neq 0)$' by 'T_{rs} for $r < s$'

* The upper limit of the third summation in equation (46) should be $(s - 3)$.

p12 equation (38) - replace $m = 1$ by $m - 1$ as the upper limit of the second summation

equation (39) - replace s by ∞ as the upper limit of the third summation

line after equation (42) - replace 'S_{r2} are zero' by 'S_{rs} are zero for $s < (r + 3)$'. Replace T_{0s} by 'T_{rs} for $r < s$'

equation (43) - replace $r + s - 1$ by $s - 1$ as the upper limit of the summation

p13 third line after equation (45) - replace infinite by infinity

p14 third line of section 4.2 - replace 'each matrix S_{r2} ,' by ', for $s < (r + 3)$, each matrix S_{rs} '

equation (46) - replace s by $s - 3$ as the upper limit of the third summation

equation (47) - replace s by ∞ as the upper limit of the third summation; and replace $\log^2(v\tau)$ by $\log^r(v\tau)$

p15 equation (51) - replace s by ∞ as the upper limit of the last summation

p17 line 8 - replace (14) by (16)

second line after equation (59) - replace 'in' by 'is', and 'had' by 'hand'

p29 first line - replace 'Wang' by 'Wong'

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1 INTRODUCTION

Traditionally the flutter problem has been treated as the determination of the stability of a system of linear differential equations. Certain authors[1-4] have however been aware of the fact that a truer representation of the physical system, for small perturbations from an equilibrium state, is a system of linear integro-differential equations, and have made some attempt to assess its significance. One way of doing this[1,3,4] is to approximate to the system of integro-differential equations by a higher order system of differential equations but this involves an approximation to the indicial aerodynamic matrix which loses some of its significant characteristics.

Stability, as the flutter investigator, or indeed the worker in the field of active control or aircraft stability, understands it, is a property of systems of differential equations. When one comes to integro-differential equations one has to redefine the concept of stability if it is going to be used. It is possible, for example, as we shall see subsequently, to have such a system, whose response to an instantaneous excitation of any sort always ultimately dies away, and which yet will respond to a certain non-instantaneous excitation, which ultimately becomes zero, with a response which does not tend to zero. This is a consequence of the hereditary nature of such systems. Do we therefore say that such a system is asymptotically stable if the response to any instantaneous excitation ultimately tends to zero, or do we require there to be no excitation ultimately tending to zero which will give a response which does not ultimately tend to zero? The precise definition of stability thus indicates a basic difference between the differential equation and integro-differential equation models. The approximation of Refs 1, 3 and 4, while providing a much improved approximation to the indicial aerodynamic matrix over that implicit in the traditional approach, does eliminate this basic difference.

The only author to consider the exact solution of the linear integro-differential equations of flutter theory is Milne[2,6]. The present study is a contribution to the same subject from a somewhat different viewpoint.

This paper is written as a companion to papers by Lawrence and Jackson[3], and Woodcock and Lawrence[4]. Originally it was intended that these two papers should be combined and issued as an ARC R & M. The present note is largely a basic study which it was planned also to include in the R & M.

2 EQUATIONS OF MOTION

For small perturbations the free motion of a deforming lifting surface, in subsonic flow is given by the solution of the matrix integro-differential equation[5].

$$(A - A_1)\frac{d^2q}{d\tau^2} + Eq = v^2 \int_0^\tau K(\tau - \tau_0)\frac{dq(\tau_0)}{d\tau_0}\,d\tau_0 \tag{1}$$

where $(A - A_1)$ is the structural inertia matrix
E is the structural stiffness matrix
A_1 is the aerodynamic inertia matrix
$K(\tau)$ is the indicial aerodynamic matrix
$q(\tau)$ is the column matrix of the generalised coordinates.

Structural damping has been neglected but, if desired, one can approximately* take account of it by including a term $D\frac{dq}{d\tau}$ on the left hand side, where D is the structural damping matrix. The indicial aerodynamic matrix $K(\tau)$, which results from a displacement with time dependence $H(\tau)$, has the form

$$K(\tau) = A_0\frac{\delta(\tau)}{v} - \frac{A_1\delta'(\tau)}{v^2} + \left\{K_\sigma + G(v\tau)\right\}H(\tau) \tag{2}$$

where $H(\tau)$, $\delta(\tau)$ are the right-handed Heaviside step and Dirac delta functions (*cf* Appendix B); A_0, A_1, K_σ are constant matrices; and $G(v\tau)$ has an asymptotic expansion of the form** (*cf* Refs 2 and 10)

$$G(v\tau) \sim \sum_{s=0}^{\infty}\frac{(-)^{s+1}s!}{(v\tau)^{s+1}}N_s . \tag{3}$$

Thus K_σ is the ultimate value, for large time, of the indicial aerodynamic matrix. If $q = \tilde{q}e^{i\omega\tau}$, where ω is real, and τ is infinitely large (*ie* simple harmonic motion of infinite duration) the coefficient of q on the right hand side of (1) becomes the oscillatory aerodynamic matrix $v^2\left\{A_1\frac{\omega^2}{v^2} - \frac{i\omega}{v}B\left(\frac{\omega}{v}\right) - C\left(\frac{\omega}{v}\right)\right\}$, where B and C are real for real argument. The Laplace transform $\bar{K}(p)$ of $K(\tau)$ is therefore related to these oscillatory aerodynamic matrices by

---

* The term structural damping embraces a multitude of effects: material damping, friction, backlash etc; and a truer representation, if possible, would introduce considerable complication.

** N_0 is zero except in the two-dimensional case.

$$p\bar{K}(p) = -\left\{\left(\frac{p}{v}\right)^2 A_1 + \frac{p}{v} B\left(-\frac{ip}{v}\right) + C\left(-\frac{ip}{v}\right)\right\} \tag{4}$$

where B and C are now continued analytically into the complex plane. Relationships therefore exist between the matrices appearing in (2) and the oscillatory aerodynamic matrices. Using the theorem[11] that

$$\mathscr{L}t_{p\to 0}\left\{p\bar{y}(p)\right\} = y(\infty) \tag{5}$$

where y is any function* and $\bar{y}$ its Laplace transform, we immediately see that

$$K_\sigma = -C(0) = \text{(say)} -C_0 . \tag{6}$$

Now, from (2)

$$p\bar{K}(p) = K_\sigma + \frac{p}{v}\left\{A_0 + \bar{G}\left(\frac{p}{v}\right)\right\} - \left(\frac{p}{v}\right)^2 A_1 \tag{7}$$

and so (4) can be rewritten

$$\frac{p}{v}\left\{A_0 + \bar{G}\left(\frac{p}{v}\right)\right\} = C_0 - C\left(-\frac{ip}{v}\right) - \frac{p}{v} B\left(-\frac{ip}{v}\right) . \tag{8}$$

If now we go to the limit $p = i\infty$ we get the relationships

$$A_0 + \mathscr{R}\left\{\bar{G}(i\infty)\right\} = -B(\infty) = \text{(say)} -B_\infty \tag{9}$$

$$-\mathscr{L}t_{\nu\to\infty}\left[\nu\mathscr{I}\left\{\bar{G}(i\nu)\right\}\right] = C_0 - C(\infty) = \text{(say)}\ C_0 - C_\infty . \tag{10}$$

From the work of Milne[2] we know that $\bar{G}\left(\frac{p}{v}\right)$ taken as single valued in the complex plane cut along the negative real axis, has the form (*cf* equation (3) and Appendix A)

$$\bar{G}\left(\frac{p}{v}\right) = \sum_{s=0}^{\infty}\left(\frac{p}{v}\right)^s L_s + \log\left(\frac{p}{v}\right)\sum_{s=0}^{\infty}\left(\frac{p}{v}\right)^s N_s \tag{11}$$

---

* Strictly any function which is bounded for real $\tau > 0$. We apply the theorem to the last term of (2).

where N_0 is zero except in the two-dimensional case. In the particular case of two-dimensional incompressible flow it is easily seen, from the known analytical solution (*eg* Ref 12) that $\bar{G}\left(\frac{p}{v}\right)$ has the form

$$\bar{G}\left(\frac{p}{v}\right) = R + \left(\frac{\Xi\left(\frac{p}{v}\right) - 1}{\frac{p}{v}}\right)\left(S + \frac{p}{v}\,T\right) \tag{12}$$

where R, S, T are constant matrices and $\Xi\left(\frac{p}{v}\right)$ is the Theodorsen function

$$\Xi\left(\frac{p}{v}\right) = \frac{K_1\left(\frac{p}{2v}\right)}{K_0\left(\frac{p}{2v}\right) + K_1\left(\frac{p}{2v}\right)} \tag{13}$$

which is taken as single valued in the complex plane apart from a slit along the negative real axis.

When the results of Appendix A (in particular equation (A-55)) are applied, in conjunction with (B-11), to equation (11) one obtains an expression for $G(v\tau)$ which is the asymptotic expansion given in equation (3) plus an infinite number of terms involving the δ function and its derivatives. Thus we obtain

$$\begin{aligned} G(v\tau) = \sum_{s=0}^{\infty} \frac{(-)^{s+1} s!\,N_s}{(v\tau)^{s+1}} H(\tau) + \Bigg[&\left\{\frac{L_0}{v}\,\delta(\tau) + \frac{L_1}{v^2}\,\delta'(\tau) + \ldots\right\} \\ &- (\gamma + \log v\tau)\left\{\frac{N_0}{v}\,\delta(\tau) + \frac{N_1}{v^2}\,\delta'(\tau) + \ldots\right\} \\ &- \frac{1}{v\tau}\left\{\frac{2N_1}{v}\,\delta(\tau) + \ldots\right\} + \ldots \Bigg] . \end{aligned} \qquad \ldots\ldots(14)$$

As pointed out in Appendix B such series as those enclosed in the $\{\ \}$ in (14) represent functions which are transcendentally small compared with the negative power of τ . The same is clearly true of $\log \tau$ or $\tau^{-m} \times$ such a series. Thus the term in the square brackets in (14), which we will call $\hat{G}(v\tau)$, satisfies the condition

$$\mathcal{L}t_{\tau\to\infty}\left\{\tau^n \hat{G}(v\tau)\right\} = 0 \qquad (n \geq 0) \tag{15}$$

and so has a zero asymptotic expansion in terms of the gauge functions $(\nu\tau)^{-m}$. Any function satisfying (15) can be interpreted (*cf* Appendix B) as the sum of a function which is zero for τ greater than some finite value, a number of a decaying exponential $\exp\{-a\tau^b\}$ $(a > 0, b > 0)$, and possibly a finite number of δ functions*. There is, however, no reason why $G(\nu\tau)$ should have non-smooth behaviour at any positive τ.

One further point to note is that the aerodynamic (or virtual) inertia matrix A, is zero if the flow is compressible. Finally we write the equation of motion (1) in the following alternative form, using (2),

$$A\frac{d^2q}{d\tau^2} - A_0\nu\frac{dq}{d\tau} + \left(E - K_\sigma\nu^2\right)q = \nu^2\int_0^\tau G(\nu\tau - \nu\tau_0)\frac{dq(\tau_0)}{d\tau_0}d\tau_0 . \qquad (16)$$

3 APPROXIMATIONS TO THE EQUATION OF MOTION

The most common approximation is to assume that the function G, in the expression for the indicial aerodynamic matrix (equation (2)), is zero and that the constant matrices A_0, K_σ have the values $-B(\nu)$, $-C(\nu)$ respectively where ν is an assumed value of the frequency parameter. Thus instead of equations (6) and (9) the approximations

$$K_\sigma = -C(\nu) \qquad (17)$$

$$A_0 = -B(\nu) \qquad (18)$$

$$\bar{G}\left(\frac{p}{\nu}\right) = 0 . \qquad (19)$$

are made. The purpose of this approximation is to obtain a differential equation which has a solution which is identical, for large time, with a solution of the original equation ((1) or (16)) in the particular case when the latter solution ultimately becomes sinusoidal with frequency parameter ν. This differential equation has the form

$$A\frac{d^2q}{d\tau^2} + \nu B(\nu)\frac{dq}{d\tau} + \left(\nu^2 C(\nu) + E\right)q = 0 . \qquad (20)$$

* This expression, hereinafter, refers to a selection from $\delta(\tau)$ and all the $\delta^{(n)}(\tau)$.

A rather better approximation, due to Richardson[1] is to approximate to G in equation (2) by a finite power series multiplied by a decaying exponential term

$$G(v\tau) \approx e^{-p_0 v\tau} \sum_{r=0}^{m-1} \frac{K_r (p_0 v\tau)^r}{r!} \quad . \tag{21}$$

Here m is a positive integer and p_0 a positive scalar. In the two-dimensional incompressible case, for the modes of heave and pitch, an established good approximation to G (see Lomax[9]) has the temporal behaviour

$$(e^{-0.09v\tau} + 2e^{-0.6v\tau}) \quad . \tag{22}$$

This suggests that a suitable choice for p_0 in the approximation (21) lies in the range 0.09 → 0.6. With this approximation (equation (21))

$$\frac{1}{v} \bar{G}\left(\frac{p}{v}\right) = \sum_{r=0}^{m-1} \frac{K_r (p_0 v)^r}{(p + p_0 v)^{r+1}} \quad . \tag{23}$$

It will be noticed, comparing equation (3) or (14) with (21), or (11) with (23), that this approximation to G (and hence to the indicial aerodynamic matrix) does not have the right behaviour as τ tends to ∞ ; nor, of course, does its transform have the right behaviour as p tends to nought, or $-p_0 v$. A suitable procedure for the determination of the K_r coefficients is described in Ref 3. In particular from equations (9), (10) and (23)

$$A_0 = -B_\infty \tag{24}$$

$$K_0 = C_0 - C_\infty \quad . \tag{25}$$

In the applications of this approximation, in Refs 3 and 4, the relationship (25) was not used, it being preferred to give equal weight to the values of the oscillatory aerodynamic matrices, at a set of frequency parameters, in the determination of the K_r coefficients.

4 THE NATURE OF THE SOLUTION

4.1 General

Taking the Laplace transform of equation (1) or (16) we obtain the characteristic equation which can be written in the two equivalent forms

$$\left|(A - A_1)p^2 + E - v^2 p\bar{K}(p)\right| = 0 \tag{26}$$

and

$$\left|Ap^2 - A_0 pv + \left(E - K_\sigma v^2\right) - pv\bar{G}\left(\frac{p}{v}\right)\right| = 0 . \tag{27}$$

The number of roots of this equation is not necessarily equal to $2n$, where n is the number of degrees of freedom, as it would be if the equation of motion were a second order differential equation. Consider, for example, the particular case of two-dimensional incompressible flow. Substituting equation (12) in equation (27) then gives

$$\left|v^2\left\{A\left(\frac{p}{v}\right)^2 - (A_0 + R)\,\frac{p}{v} + \frac{E}{v^2} - K_\sigma - \left(\frac{p}{v}\,T + S\right)\left(\Xi\left(\frac{p}{v}\right) - 1\right)\right\}\right| = 0 . \tag{28}$$

This function is single valued in the domain of the complex plane bounded by two circles around the origin, of very large and very small radius, and two lines just above and just below the negative real axis - *ie* the whole complex plane apart from a slit along the negative real axis. On the boundaries of this domain $\Xi\left(\frac{p}{v}\right)$ has the following values therefore

(i) on the large circle $\Xi = 0.5$;

(ii) on the small circle $\Xi = 1.0$;

(iii) on the edge of the slit the $\mathcal{R}(\Xi)$ is always positive; and the $\mathcal{I}(\Xi)$ is always negative above the slit and positive below the slit except for the two ends where it is zero (*cf* Table 1).

If then we apply the principle of the argument (Copson[8]) to the function (28) as we traverse the boundary of this domain we find, for the case of one degree of freedom ($n = 1$) that the function can have either two or three zeroes within the domain. Consequently for n degrees of freedom the number of zeroes can be anything between $2n$ and $3n$.

There may in addition be some negative real roots. Milne[2] argued that, because the imaginary part of the function which is equated to zero in the

characteristic equation is not single valued at points on the negative real axis then the imaginary part cannot become zero at a point on the negative real axis. However, writing Ξ^r at a point $\left(\frac{p}{v} = -x\right)$ on the negative real axis as

$$\Xi^r(-x) = \zeta_r(x) \pm i\eta_r(x) \tag{29}$$

then the characteristic equation (28), which has the form

$$\sum_{r=0}^{n} a_r\left(\frac{p}{v}\right) \Xi^r \left(\frac{p}{v}\right) = 0 \tag{30}$$

is satisfied, at such a point, if

$$\left.\begin{aligned} a_0(-x) + \sum_{r=1}^{n} a_r(-x)\zeta_r(x) &= 0 \\ \text{and} \qquad \sum_{r=1}^{n} a_r(-x)\eta_r(x) &= 0 \end{aligned}\right\} \tag{31}$$

both have the same positive real root (for x). There is no apparent reason why this could not be so and we therefore cannot rule out the possibility of there being a negative real root.

It is not as easy to determine limits on the number of roots of the characteristic equation for the general case (not two-dimensional incompressible) but the above example makes it obvious that there may be more than 2n.

With the use of (11) the characteristic equation (27) can be written

$$\left| v^2\left\{(A - L_1)\left(\frac{p}{v}\right)^2 - (A_0 + L_0)\frac{p}{v} + \left(\frac{E}{v^2} - K_\sigma\right) - \sum_{s=2}^{\infty} L_s\left(\frac{p}{v}\right)^{s+1} - \frac{p}{v}\log\left(\frac{p}{v}\right) \sum_{s=0}^{\infty} N_s\left(\frac{p}{v}\right)^s\right\}\right| = 0$$

.....(32)

At first sight one may think this equation has an infinite number of roots, but, as we have just seen this is not necessarily so. One surmises that the number of roots will always be finite and equal to or a little greater than 2n. Knowledge of these roots, and the associated characteristic vectors, will enable us to derive an expression for the inverse of the characteristic matrix

$$v^2 M\left(\frac{p}{v}\right) \equiv Ap^2 - A_0 pv + E - K_\sigma v^2 - pv\bar{G}\left(\frac{p}{v}\right) \tag{33}$$

and hence the response of the system to an arbitrary impulse. We will assume all the roots of (32) are distinct. Let these be λ_i $(i = 1 \to k)$; and α_i', β_i the associated left hand and right hand characteristic vectors. Then*

$$\left.\begin{aligned} \operatorname*{Limit}_{p \to \lambda_i}\left\{\left(\frac{p - \lambda_i}{v}\right) M^{-1}\left(\frac{p}{v}\right)\right\} &= \frac{\beta_i \alpha_i'}{\alpha_i'\left(\frac{dM(u)}{du}\right)_{u = \frac{\lambda_i}{v}} \beta_i} \\ &= W_i \quad \text{(say)} \end{aligned}\right\} \tag{34}$$

and so M^{-1} can be written, remembering (11), in the form

$$M^{-1}\left(\frac{p}{v}\right) = v \sum_{i=1}^{k} \frac{W_i}{(p - \lambda_i)} + v^2 \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \left(\frac{p}{v}\right)^{r+s} \log^s\left(\frac{p}{v}\right) T_{rs}(v) \tag{35}$$

provided $(E - K_\sigma v^2)$ is not singular. In these equations the α_i, β_i, λ_i (and hence W_i) will either be real or occur in complex conjugate pairs; the T_{rs} matrices will be real. In particular

$$T_{00} = (E - K_\sigma v^2)^{-1} + \frac{1}{v} \sum_{i=1}^{k} \frac{W_i}{\lambda_i} \tag{36}$$

and

$$T_{01} = (E - K_\sigma v^2)^{-1} N_0 (E - K_\sigma v^2)^{-1} . \tag{37}$$

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* $$W_i = \left(\text{adjoint of } M \times \frac{d|M|}{du}\right)_{u = \frac{\lambda_i}{v}}$$

We notice also that all the T_{0s} $(s \neq 0)$ are zero if N_0 is zero. In Appendix A an expression, (A-37), was obtained for the inverse transforms of $\log^m p$. Taking this in the form

$$\theta_m(\tau) = (-)^m \left[\left\{ \sum_{v=0}^{m} \binom{m}{v} w_v(0) \log^{m-v}\tau \right\} \delta(\tau) + m \left\{ \sum_{v=0}^{m-1} \binom{m-1}{v} w_v(0) \log^{m-v-1}\tau \right\} \frac{H(\tau)}{\tau} \right], \quad \ldots\ldots(38)$$

where calculated values of the coefficients $w_v(0)$ are given in Table 3; it follows that, for $\tau > 0$ and $\left(E - K_\sigma \nu^2\right)$ non-singular, the inverse transform of $M^{-1}\left(\frac{p}{v}\right)$ has the form, using (B-11)

$$\left.\begin{aligned} \mathscr{L}^{-1}\left\{M^{-1}\left(\frac{p}{v}\right)\right\} &= v \sum_{i=1}^{k} W_i e^{\lambda_i \tau} + v^3 \sum_{s=2}^{\infty} \sum_{r=0}^{s} \frac{\log^r(v\tau)}{(v\tau)^{r+s}} S_{rs}(v) \\ &= \text{(say)}\ X_0(\tau) \end{aligned}\right\} \quad (39)$$

where the S_{rs} matrices are real. In particular we note that

$$S_{02} = T_{01} \quad (40)$$

$$S_{12} = 4T_{02} \quad (41)$$

$$S_{03} = (4\gamma - 6)T_{02} - 2T_{11} \quad (42)$$

etc

and that all the S_{r2} are zero if N_0 (and hence - each T_{0s}) is zero. The general form for the S_{rs} is

$$S_{rs} = \sum_{u=1}^{r+s-1} \alpha_u^{(rs)} T_{u-1,r+s-u} \quad (43)$$

where the coefficients $\alpha_u^{(rs)}$ can be determined from (38) and (B-11).

It will be noticed that an infinite series of terms in the δ function and its derivatives has apparently been omitted from (39). The coefficients in this series, which by (B-11) are constants, cannot easily be evaluated since they are the limit ($\tau \to 0$) of rather complicated sums of terms each of which individually becomes infinite at the limit. We assume that each coefficient has an appropriate finite value*. Then, as shown in Appendix B, the series omitted from (39) represents a function which is either zero for τ greater than some finite value, or else decays, as $\tau \to \infty$, as a simple exponential $e^{-a\tau}$ or more rapidly. There is no reason why X_0 should have any non-smooth behaviour at finite τ, and we have already included in (39) all the possible simple exponential terms. Consequently the function omitted, if it is not a null function, must be a function which decays more rapidly, with increasing τ, than any of the terms we have given in the above expression for X_0 (equation (39)).

An arbitrary instantaneous excitation at $\tau = 0+$ can be taken to be

$$\delta(\tau)f_0 + \delta^{(1)}(\tau)f_1 + \delta^{(2)}(\tau)f_2 + \dots$$

where $f_0, f_1, \dots$ are arbitrary real constant column matrices; and so the free motion of the system (*ie* the motion after the disturbance) is given by

$$q = \frac{1}{v^2}\sum_{j=0}^{\infty} X_j f_j \tag{44}$$

where

$$X_j = \frac{d^j X_0}{d\tau^j} \tag{45}$$

and X_0 is given by equation (39).

Milne has derived the same form, for the fundamental solution X_0, in Ref 2 by a different method. The infinite of arbitrary constants (the elements

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* Thus the coefficient of $\delta(t)$ contains $\frac{1}{\tau} \times$ a power series in $(\log \tau/\tau)$, and in the simplest case, for example, we can make the identification

$$\mathscr{L}t_{\tau\to 0}\left\{\frac{1}{\tau}\sum_{T=0}^{\infty}\left(\frac{\log \tau}{\tau}\right)^r\right\} = \mathscr{L}t_{\tau\to 0}\left(\frac{1}{\tau - \log \tau}\right) = 0 .$$

of the matrices f_j) arises from the fact that the free motion subsequent to a given instant depends not only on the velocity and displacements at that instant (as with an instantaneous dynamical system) but on the whole history of the motion previous to that instant. For large time the $\frac{1}{\tau^2}$ terms in the above solution will be dominant*; for intermediate times the exponential terms need also to be taken into account, and indeed Milne[6] has suggested, with some supporting evidence, that in practice the leading S_{rs} coefficients (*eg* (39)) are small compared with the W_i and so there will be a large range of time for which the motion is predominantly exponential. For very small time the form of solution that we have obtained is of course inadequate.

4.2 Particular cases

At this point we will note certain features of the fundamental solution which appears in particular cases. Firstly if the flow is three-dimensional the matrix N_0 , and hence each matrix S_{r2} , is zero and so the fundamental solution has the form

$$X_0 = v \sum_{i=1}^{k} W_i e^{\lambda_i \tau} + v^3 \sum_{s=3}^{\infty} \sum_{r=0}^{s} \log^r (v\tau) \frac{S_{rs}(v)}{(v\tau)^{r+s}} \qquad (46)$$

in which the dominant term for large time is of $0\left(\frac{1}{\tau^3}\right)$. Secondly if the matrix $\left(E - K_\sigma v^2\right)$ is singular, that is if the characteristic equation (32) has a zero root $\lambda_1 = 0$, then the dominant non-exponential term, apart from the constant term, is of $0\left(\frac{1}{\tau}\right)$ (*cf* Ref 6) and so the fundamental solution has the form

$$X_0 = v\left(W_1 + \sum_{i=2}^{k} W_i e^{\lambda_i \tau}\right) + v \sum_{s=1}^{\infty} \sum_{r=0}^{s} \log^2 (v\tau) \frac{U_{rs}(v)}{(v\tau)^{r+s}} \quad . \qquad (47)$$

This can be simply demonstrated for the one degree of freedom system by expanding $\left(\frac{p}{v}\right)M^{-1}$ in the form of equation (35), obtaining its inverse transform using the results of Appendix A and hence obtaining the inverse transform of M^{-1} . The two-dimensional case, when the characteristic equation has a zero root, is not as simple.

* Assuming all the λ_i have negative real parts.

We take as our independent variable $\frac{p \log(p/v)}{v - p}$, which is a monotone function of p , and note that

$$\left.\begin{aligned} \operatorname*{Limit}_{p\to\lambda_i}\left[\left\{\frac{p\log\left(\frac{p}{v}\right)}{v-p}-\frac{\lambda_i\log\left(\frac{\lambda_i}{v}\right)}{v-\lambda_i}\right\}M^{-1}\left(\frac{p}{v}\right)\right] &= \frac{\beta_i\alpha_i'}{\alpha_i'\left(\dfrac{dM(u)}{d\left(\frac{u\log u}{1-u}\right)}\right)_{u=\frac{\lambda_i}{v}}\beta_i} \\ &= W_i^* \quad \text{(say)} \end{aligned}\right\} \tag{48}$$

where α_i', β_i are the left hand and right hand characteristic vectors associated with the root $p = \lambda_i$. If $i = 1$ denotes the zero root, *ie* $\lambda_1 = 0$, then for $i \neq 1$ the W_i^* are related to the W_i (equation (34)) *viz*

$$W_i^* = \operatorname*{Limit}_{p\to\lambda_i}\left\{\frac{vW_i}{p-\lambda_i}\left(\frac{p\log\left(\frac{p}{v}\right)}{v-p}-\frac{\lambda_i\log\left(\frac{\lambda_i}{v}\right)}{v-\lambda_i}\right)\right\} \tag{49}$$

Consequently the inverse of M can be expressed from* (48) and (49) in the form

$$M^{-1}\left(\frac{p}{v}\right) = \frac{W_1^*\left(1-\frac{p}{v}\right)}{\frac{p}{v}\log\frac{p}{v}} + v\sum_{i=2}^{k}\frac{W_i}{(p-\lambda_i)} + v^2\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left(\frac{p}{v}\right)^{r+s}\log^s\left(\frac{p}{v}\right)T_{rs}^*(v) \quad . \tag{50}$$

This gives a fundamental solution (the inverse transform of M^{-1}) of the form (see Appendix A)

$$\left.\begin{aligned} X_0 = &- vW_1^*\left[\sum_{u=0}^{\infty}(-)^u\frac{w_u(\chi)}{(\chi+\log v\tau)^{u+1}}+\frac{1}{v\tau}\sum_{u=0}^{\infty}(-)^u\frac{(u+1)w_u(\chi)}{(\chi+\log v\tau)^{u+2}}\right] \\ &+ v\sum_{i=2}^{k}W_ie^{\lambda_i\tau}+v^3\sum_{s=2}^{\infty}\sum_{r=0}^{s}\log^r(v\tau)\frac{S_{rs}^*(v)}{(v\tau)^{r+s}} \qquad \text{for } \tau > 0 \end{aligned}\right\} \tag{51}$$

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* We are still assuming all the λ_i are distinct. In addition we assume that no $\lambda_i = v$, otherwise equation (50) will require some modification.

for the present case when the flow is two-dimensional and the characteristic equation has a zero root. The w_u coefficients are given in Table 3, having been determined from equations (A-34) and (A-35). The value of χ is arbitrary, but, as explained in Appendix A, for any particular value of $\nu\tau$, it has to be suitably chosen so that a sufficiently accurate 'sum' can be obtained from the asymptotic series. Thus, for example, for $\nu\tau = 1$, with $\chi = -3\gamma$, where γ is Euler's constant, the first series can be evaluated to about three significant figures; but with $\chi = 3\gamma$ not a single significant figure is obtained (*cf* Appendix A). For τ large, from equations (A-45) and (A-53) or otherwise

$$\sum_{u=0}^{\infty} (-)^u \frac{w_u(\chi)}{(\chi + \log \nu\tau)^{u+1}} \approx \frac{1}{\log \nu\tau} . \tag{52}$$

Similarly

$$\sum_{u=0}^{\infty} (-)^u \frac{(u+1)w_u(\chi)}{(\chi + \log \nu\tau)^{u+2}} \approx \frac{1}{\log^2 \nu\tau} \quad \text{for } \tau \text{ large} \tag{53}$$

Consequently in this case (two-dimensional, characteristic equation with zero root) the dominant term for large time is $O\left(\frac{1}{\log \tau}\right)$ provided all the λ_i $(i = 2 \to k)$ have negative real parts. The T^*_{rs} matrices appearing in (50) are given by rather complicated expressions. In particular

$$\left.\begin{aligned} T^*_{00} = {} & \left\{ \nu^2 N_0 W^*_1 + (E - K_\sigma \nu^2) \sum_{i=2}^{k} \frac{W_i}{\lambda_i} \right\}^{-1} \times \\ & \times \sum_{i=2}^{k} \left\{ \frac{\nu}{\lambda_i} \left(W_i + W^*_1 N_0 W_i + W_i N_0 W^*_1 \right) \right. \\ & \left. + \sum_{j=2}^{k} \frac{W_i \left(E - K_\sigma \nu^2\right) W_j}{\lambda_i \lambda_j} \right\} . \end{aligned}\right\} \tag{54}$$

The relationship between the S^*_{rs} matrices and the T^*_{rs} matrices is the same as that between the S_{rs} and the T_{rs} (equations (40) to (43)). It is perhaps a

little surprising to find (*cf* equation (45))

$$\mathscr{L}t_{\tau\to\infty}\ (X_j) = 0 \qquad \text{for all } j \tag{55}$$

in a case where the characteristic equation has a zero root. One would expect that there would be excitation, which ultimately died away, which would produce a response

$$q = \beta H(\tau) \tag{56}$$

That is there would be a means of producing a 'free' motion which was a constant displacement. However if we surmise the displacement (56) then from equation (14) the necessary excitation is

$$\left(A\delta''(\tau) - A_0 v\delta'(\tau) - v^2 G(v\tau) + \left(E - K_\sigma v^2\right)\right)\beta H(\tau)\ . \tag{57}$$

Since

$$\mathscr{L}t_{\tau\to\infty}\ G(v\tau) = 0 \tag{58}$$

then, as τ tends to ∞, this force tends to

$$\left(E - K_\sigma v^2\right)\beta\ . \tag{59}$$

In the case under consideration $\left(E - K_\sigma v^2\right)$ is singular and so the excitation force (57) will ultimately tend to zero if β in the right had eigenvector β_1 of $\left(E - K_\sigma v^2\right)$. A 'free' motion of constant displacement is therefore possible. Such motion cannot however be achieved by an instantaneous excitation for we have from (14)

$$\left.\begin{aligned} G(v\tau)\beta_1 &= \sum_{s=0}^{\infty} \frac{(-)^{s+1} s!}{(v\tau)^{s+1}} N_s \beta_1 H(\tau) \\ &\quad + \text{an impulsive term}\ . \end{aligned}\right\} \tag{60}$$

For this reason we have used inverted commas in designating such motion as 'free' motion. In the three-dimensional case, when the characteristic equation has a zero root, it follows from (47) that an impulse $\delta(\tau)f_0$ will produce a response which is ultimately a constant displacement but here again a response of the form (56) can only be achieved by an excitation which is not entirely impulsive.

4.3 Discussion

To recapitulate we see that we have obtained four forms for the fundamental solution X_0. If the matrix $(E - K_\sigma v^2)$ is not singular, that is if the characteristic equation does not have a zero root, X_0 is given by equation (39) or (46) according as the flow is two-dimensional or three-dimensional. When $(E - K_\sigma v^2)$ is singular, however, we have the modified forms (51) and (47) for two- and three-dimensional flow respectively. In every case we have assumed that the roots of the characteristic equation are distinct. Cases of multiple roots will produce some further modification to the fundamental solution. The free motion of the system, for $\tau > 0$, is then given by equation (44). It will be noticed that in each case this solution involves constituents $\beta_i e^{\lambda_i t}$, though there are other constituents as well. One expects this, but it should be noted that the number of these exponential terms may well be more than $2n$, where n is the number of degrees of freedom, and also that the right hand characteristic vectors β_i have no orthogonal relationship with the left hand characteristic vector α_i'. If $\bar{G}\left(\frac{p}{v}\right)$ were zero, as is assumed in the first approximation considered in section 3 (equation (19)), then (*cf* equation (27)) these vectors would satisfy the biorthogonal relationships

$$\begin{bmatrix} \alpha_i' & \lambda_i \alpha_i' \end{bmatrix} \begin{bmatrix} K_\sigma v^2 - E & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \beta_j \\ \lambda_j \beta_j \end{bmatrix} = 0 \qquad (i \neq j) \qquad (61)$$

$$\begin{bmatrix} \alpha_i' & \lambda_i \alpha_i' \end{bmatrix} \begin{bmatrix} 0 & K_\sigma v^2 - E \\ K_\sigma v^2 - E & A_0 v \end{bmatrix} \begin{bmatrix} \beta_j \\ \lambda_j \beta_j \end{bmatrix} = 0 \qquad (i \neq j) \qquad (62)$$

but there is nothing corresponding when no approximation is made.

5 CONCLUDING REMARKS

It follows from the forms of fundamental solution derived in the previous section (equations (39), (46), (47) and (51)) that an aeroelastic system will be stable, in the sense that the response to any impulsive excitation ultimately dies away to zero, for small perturbations in subsonic flow provided all the roots of the characteristic equation ((26), (27)) have negative real parts*.

* We have seen that in two-dimensional flow we can still have stability in this sense when the characteristic equation has a zero root.

Thus the traditional UK method of flutter investigation with lined-up frequency parameter[3] is completely adequate for determining such stability; for it determines any roots with zero real part correctly and so by a survey of the speed range finds whether any root becomes unstable. There is also some evidence[4] that it determines complex roots with a fair degree of accuracy even when the real part is negative and relatively large (compared with the imaginary part).

Often one wishes to know not only the critical flutter speed of an aircraft, if there is one in the flight domain, but also how quickly the effect of a disturbance will become insignificant. If Milne's[6] supposition that, in equation (39) for example, the leading S_{rs} coefficients are usually small compared with the coefficients, W_i, of the exponential terms, then the traditional UK approach should also provide this information reasonably well. The characteristic value λ_i which has the least negative real part given the required information.

However when the non-exponential terms in the fundamental solution are of significant size at intermediate times one would have to consider also at least the dominant one of such terms, which in the two-dimensional case is (see equations (37), (39) and (40))

$$\frac{v}{\tau^2}\left(E - K_\sigma v^2\right)^{-1} N_0 \left(E - K_\sigma v^2\right)^{-1} \tag{63}$$

and in the three-dimensional case* (see equations (42) and (46)) is

$$-\frac{2}{\tau^3}\left(E - K_\sigma v^2\right)^{-1} N_1 \left(E - K_\sigma v^2\right)^{-1} . \tag{64}$$

If one determined the appropriate expansion of the oscillatory aerodynamic matrix $C(\nu) + i\nu B(\nu)$ for small ν then (*cf* equations (8) and (11)) this would provide the required matrix N_0 or N_1 .

In this note we have only considered the free behaviour of our aeroelastic system. The fundamental solution $X_0(\tau)$ could in principle be also used to obtain the response to any excitation by evaluating the *faltung* integral of X_0

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* Since N_0 is zero so is T_{02} and it is found then that

$$T_{11} = \left(E - K_\sigma v^2\right)^{-1} N_1 \left(E - K_\sigma v^2\right)^{-1}$$

in a form convenient for use at small τ which we have not obtained. It would probably therefore be advisable to determine first the Laplace transform of the response and then invert that.

Appendix A

LAPLACE TRANSFORMS OF $\log^m t$ AND INVERSE LAPLACE TRANSFORMS OF $\log^m p$

The Gamma function can be defined by the integral[7]

$$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du \quad . \tag{A-1}$$

Differentiating this n times with respect to z gives

$$\Gamma^{(n)}(z) = \int_0^\infty e^{-u} \log^n u \; u^{z-1} du \tag{A-2}$$

and, putting $z = 1$, we have

$$\Gamma^{(n)}(1) = \int_0^\infty e^{-u} \log^n u \; du \quad . \tag{A-3}$$

The Laplace transform of $\log^m t$, for $m \geqslant 0$, is therefore given by

$$\left.\begin{aligned} \int_0^\infty e^{-pt} \log^m t \; dt &= \frac{1}{p} \int_0^\infty e^{-u} \{\log u - \log p\}^m du \\ &= \frac{(-)^m}{p} \sum_{r=0}^{m} \binom{m}{r} (-)^r \Gamma^{(r)}(1) \log^{m-r} p \end{aligned}\right\} \quad . \tag{A-4}$$

Wong[13,16] has shown, *inter alia*, that a similar formula applies for negative powers of $\log t$. He considers the transforms of functions $f(t)$ which are locally integrable and satisfy

$$f(t) \sim \log^{-n} t \qquad \text{as } t \to \infty \tag{A-5}$$

where n is positive, and so he proves in particular that the Laplace transform

of $\log^{-n}t$ has the asymptotic expansion*

$$\int_0^\infty e^{-pt} \log^{-n}t \, dt \sim \frac{(-)^n}{p} \sum_{r=0}^{\infty} \binom{-n}{r} (-)^r \frac{\Gamma^{(r)}(1)}{\log^{n+r}p} \qquad \text{as } p \to 0 . \quad \text{(A-6)}$$

$\left(\text{This is an expansion using the sequence of gauge functions } \frac{1}{p \log^n p}, \frac{1}{p \log^{n+1} p}, \ldots\right)$.

The values of the derivatives of $\Gamma(z)$ at $z = 1$ can be obtained from the well known series[7] for $\frac{d}{dz}\left\{\log \Gamma(z)\right\}$, *ie*

$$\Psi_1(z) \equiv \frac{\Gamma^{(1)}(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z}\right) \qquad \text{(A-7)}$$

where γ (= 0.577215....) in Euler's constant.

$\Psi_1(z)$ and its derivatives

$$\Psi_r(z) \equiv \frac{d^r}{dz^r}\left\{\log \Gamma(z)\right\} = \Gamma(r)(-)^r \sum_{n=0}^{\infty} \frac{1}{(n+z)^r} \qquad \text{(A-8)}$$

are the polygamma functions. A simple general expression for the $\Gamma^{(n)}(z)$ cannot be written down. However, differentiating $\Gamma^{(1)}(z) \left\{= \Psi_1(z)\Gamma(z)\right\}$ $(n - 1)$ times, using Leibniz's theorem, we obtain the recurrence relationship

$$\Gamma^{(n)}(z) = \sum_{r=1}^{n} \binom{n-1}{r-1} \Psi_r(z)\Gamma^{(n-r)}(z) , \qquad \text{(A-9)}$$

and so, in particular

$$\Gamma^{(n)}(1) = \sum_{r=1}^{n} \frac{\Gamma(n)}{\Gamma(n-r+1)} \rho_r \Gamma^{(n-r)}(1) \qquad \text{(A-10)}$$

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* The meaning of this is that the sum of a finite number of terms (N) differs from the required function by $\frac{1}{p} O\left(\frac{1}{\log^{n+N}p}\right)$ for every integer $N \geq 0$.

where $$\rho_1 = \frac{\Psi_1(1)}{\Gamma(1)} = -\gamma \tag{A-11}$$

and

$$\rho_r = \frac{\Psi_r(1)}{\Gamma(r)} = (-)^r \sum_{s=1}^{\infty} \frac{1}{s^r} \qquad (r \geq 2)\ . \tag{A-12}$$

A more general relationship than (A-10), involving the operator $(D + \chi)$, where $D = \frac{d}{dz}$, will however be useful to us. To obtain this we note that Leibniz's theorem can be generalised to the form

$$(D + \alpha + \beta)^n(uv) = \sum_{r=0}^{n} \binom{n}{r} (D + \alpha)^r(u)(D + \beta)^{n-r}(v) \tag{A-13}$$

of which a particular case is

$$(D + \chi)^{n-1}\left\{\Psi_1(z)\Gamma(z)\right\} = \sum_{r=0}^{n-1} \binom{n-1}{r} \Psi_{r+1}(z)(D + \chi)^{n-1-r}\left\{\Gamma(z)\right\} \tag{A-14}$$

where χ is arbitrary.

Thus we have

$$(D + \chi)^n\{\Gamma(z)\} = \sum_{r=1}^{n} \binom{n-1}{r-1} \Psi_r(z)(D + \chi)^{n-r}\{\Gamma(z)\} + \chi(D + \chi)^{n-1}\{\Gamma(z)\} \tag{A-15}$$

and putting $z = 1$ and writing

$$\xi_n(\chi) = \left[(D + \chi)^n\{\Gamma(z)\}\right]_{z=1} = \sum_{s=0}^{n} \binom{n}{n-s} \chi^s \Gamma^{(n-s)}(1) \tag{A-16}$$

this gives

$$\left.\begin{aligned} \xi_n(\chi) &= \sum_{r=1}^{n} \frac{\Gamma(n)}{\Gamma(n-r+1)} \sigma_r \xi_{n-r}(\chi) & (n \geq 1) \\ &= 1 & (n = 0) \end{aligned}\right\} \quad \text{(A-17)}$$

where
$$\left\{\begin{aligned} \sigma_1 &= \rho_1 + \chi = \chi - \gamma & & \text{(A-18)} \\ \sigma_r &= \rho_r & (r \geq 2)\ . & \ \text{(A-19)} \end{aligned}\right.$$

We now require the inverse transforms of $\log^m p$ which we will denote by $\theta_m(t)$, where

$$\int_0^\infty e^{-pt}\theta_m(t)dt = \log^m p \quad . \tag{A-20}$$

From equation (A-4), for positive m , using (A-16)

$$\left.\begin{aligned} \int_0^\infty e^{-pt}(\chi + \log t)^m dt &= \frac{(-)^m}{p}\sum_{r=0}^{m}\binom{m}{r}(-)^r\Gamma^{(r)}(1)\{\log p - \chi\}^{m-r} \\ &= \frac{1}{p}\sum_{s=0}^{m}(-)^s\binom{m}{s}\log^s p\sum_{r=0}^{m-s}\Gamma^{(r)}(1)\chi^{m-s-r}\binom{m-s}{r} \\ &= \frac{1}{p}\sum_{s=0}^{m}(-)^s\binom{m}{s}\xi_{m-s}(\chi)\log^s p \quad . \end{aligned}\right\} \quad \text{(A-21)}$$

Let X be the infinite lower triangular matrix whose non-zero elements are

$$x_{ij} = \binom{i-1}{j-1}\xi_{i-j}(\chi) \qquad (i \geq j)\ . \tag{A-22}$$

Thus, in particular, *cf* equation (A-17), X has unit elements on the principal diagonal*. Its inverse will then have the form

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* If we chose $\chi = \gamma$ it would (see (A-18)) also have zero elements on the first subdiagonal.

$$X^{-1} = I + S \tag{A-23}$$

where S is a lower triangular matrix with zero elements on the principal diagonal*. We can then solve the infinite set of equations (A-21) $(m = 0 \to \infty)$ in terms of the elements s_{rs} of S. Thus

$$\frac{(-)^s \log^s p}{p} = \int_0^\infty e^{-pt}(\chi + \log t)^s dt + \sum_{v=1}^{s} s_{s+1,v}(\chi) \int_0^\infty e^{-pt}(\chi + \log t)^{v-1} dt .$$

for $s \geqslant 0$ .....(A-24)

It follows that the inverse transform of $\log^m p$, for m positive, is

$$\theta_m(t) = (-)^m \left[\left\{ (\chi + \log t)^m + \sum_{v=1}^{m} (\chi + \log t)^{v-1} s_{m+1,v}(\chi) \right\} \delta(t) \right.$$
$$\left. + \left\{ m(\chi + \log t)^{m-1} + \sum_{v=2}^{m} (v-1)(\chi + \log t)^{v-2} s_{m+1,v}(\chi) \right\} \frac{H(t)}{t} \right] .$$

.....(A-25)

At first sight there may seem to be an arbitrariness about this expression since χ is arbitrary. However it will be apparent, when we obtain expressions for the $s_{uv}(\chi)$, that this is not so. Indeed the main purpose of introducing an arbitrary χ , rather than taking it $= \gamma$ or 0, is in the subsequent derivation of $\theta_m(t)$ when m is negative.

From the definition of S , equation (A-23), it follows that its elements along a diagonal are related

$$\left. \begin{aligned} s_{uv} &= \binom{u-1}{v-1} s_{u-v+1,1} \\ &= \text{(say)} \binom{u-1}{v-1} w_{u-v}(\chi) \end{aligned} \right\} . \tag{A-26}$$

* If we chose $\chi = \gamma$ it would (see (A-18)) also have zero elements on the first subdiagonal.

The first column of $(I + S)$ is thus, using (A-23)

$$\{w_0 \quad w_1 \quad w_2 \quad \cdots\} \tag{A-27}$$

where
$$w_0 = 1 \tag{A-28}$$

and

$$w_n(\chi) = -\sum_{r=1}^{n} \binom{n}{r} w_{n-r}(\chi)\xi_r(\chi). \tag{A-29}$$

Equations (A-29) and (A-17) are however probably not the most convenient way to determine the $w_n(\chi)$. It is obvious that there is also the reciprocal relationship

$$\xi_n(\chi) = -\sum_{r=1}^{n} \binom{n}{r} \xi_{n-r}(\chi) w_r(\chi) \qquad (n \geq 1) . \tag{A-30}$$

Substituting from (A-16) in this equation and reversing the order of summation on the right hand side then gives

$$\sum_{s=0}^{n} \binom{n}{n-s} \chi^{n-s}\Gamma^{(s)}(1) = -\sum_{s=1}^{n} \binom{n}{n-s} \chi^{n-s} \sum_{r=0}^{s-1} \binom{s}{r} w_{s-r}(\chi)\Gamma^{(r)}(1) \tag{A-31}$$

ie

$$\chi^n + \sum_{s=1}^{n} \binom{n}{n-s} \chi^{n-s} \left\{ \Gamma^{(s)}(1) + \sum_{r=0}^{s-1} \binom{s}{r} w_{s-r}(\chi)\Gamma^{(r)}(1) \right\} = 0 . \tag{A-32}$$

This equation is true when n is any positive integer and so it is easily seen by induction that we must have

$$\Gamma^{(s)}(1) + \sum_{r=0}^{s-1} \binom{s}{r} w_{s-r}(\chi)\Gamma^{(r)}(1) = (-\chi)^s \tag{A-33}$$

which can be rewritten, bearing in mind (A-28), as

$$\frac{w_n(\chi)}{\Gamma(n+1)} = \frac{(-\chi)^n}{\Gamma(n+1)} - \sum_{r=1}^{n} \left(\frac{w_{n-r}(\chi)}{\Gamma(n-r+1)}\right)\left(\frac{\Gamma^{(r)}(1)}{\Gamma(r+1)}\right) . \tag{A-34}$$

Equation (A-10) gives

$$\frac{\Gamma^{(n)}(1)}{\Gamma(n+1)} = \frac{1}{n} \sum_{r=1}^{n} \rho_r \frac{\Gamma^{(n-r)}(1)}{\Gamma(n-r+1)} \tag{A-35}$$

and so these two equations, (A-34) and (A-35), provide a suitable means of successively evaluating the $\left(w_n(\chi)/\Gamma(n+1)\right)$ from the values of the ρ_r (equations (A-11) and (A-12)). We have

$$\left.\begin{aligned} \rho_r &\to (-1)^r \\ \frac{\Gamma^{(r)}(1)}{\Gamma(r+1)} &\to (-1)^r \end{aligned}\right\} . \qquad r \to \infty \tag{A-36}$$

Values of these two functions are given in Table 2. The ρ_r were evaluated, to rather more significant figures than shown in the table, using Euler's summation formula[14]. Values of the $w_r(\chi)$, calculated from (A-34), are given in Table 3 for several values of χ .

Equation (A-25), for the inverse transform of $\log^m p$, for m positive, can, using (A-26) and (A-28), be rewritten as

$$\begin{aligned} \theta_m(t) = (-)^m \Bigg[& \left\{ \sum_{v=0}^{m} \binom{m}{v} w_v(\chi)(\chi + \log t)^{m-v} \right\} \delta(t) \\ & + m \left\{ \sum_{v=0}^{m-1} \binom{m-1}{v} w_v(\chi)(\chi + \log t)^{m-v-1} \right\} \frac{H(t)}{t} \Bigg] . \end{aligned} \tag{A-37}$$

Now, substituting for $\xi_n(\chi)$ from equation (A-16) in (A-30) gives, after a little manipulation

$$\sum_{s=0}^{n} \binom{n}{s} \Gamma^{(n-s)}(1) \sum_{r=0}^{s} \binom{s}{r} \chi^{s-r} w_r(\chi) = 0 \tag{A-38}$$

while from (A-33) we have

$$\sum_{s=0}^{n} \binom{n}{s} \Gamma^{(n-s)}(1) w_s(0) = 0 \ . \tag{A-39}$$

Comparing these two equations we immediately see that we must have

$$\sum_{r=0}^{s} \binom{s}{r} \chi^{s-r} w_r(\chi) = w_s(0) \ . \tag{A-40}$$

It follows therefore that there is no arbitrariness in equation (A-37).

Let us now consider the inverse transform of $\log^m p$, when m is negative. From (A-6), using (A-16) and (A-22), since we can add asymptotic series as if they were convergent,

$$\left.\begin{aligned}
\int_0^\infty \frac{e^{-pt}}{(\chi + \log t)^n} dt &\sim \frac{(-)^n}{p} \sum_{r=0}^{\infty} \binom{-n}{r} \frac{(-)^r \Gamma^{(r)}(1)}{(\log p - \chi)^{n+r}} \qquad \text{as } p \to 0 \\
&= \frac{(-)^n}{p} \sum_{s=0}^{\infty} \frac{(-)^s \binom{-n}{s}}{\log^{n+s} p} \sum_{r=0}^{s} \binom{s}{r} \chi^{s-r} \Gamma^{(r)}(1) \\
&= \frac{(-)^n}{p} \sum_{s=0}^{\infty} \frac{(-)^s \binom{-n}{s} \xi_s(\chi)}{\log^{n+s} p} \\
&= \frac{(-)^n}{p} \sum_{r=n}^{\infty} \frac{\binom{r-1}{n-1} \xi_{r-n}(\chi)}{\log^r p} \\
&= \frac{(-)^n}{p} \sum_{r=n}^{\infty} \frac{x_{rn}}{\log^r p} \ .
\end{aligned}\right\} \tag{A-41}$$

This result can be confirmed by the use of Wang's[16] general formula for the Laplace transforms of function of the form (A-5). Consequently

$$\left.\begin{aligned}
&(-)^n \int_0^\infty \frac{e^{-pt}}{(\chi + \log t)^n}\,dt + \sum_{v=n+1}^{\infty} (-)^v s_{vn} \int_0^\infty \frac{e^{-pt}}{(\chi + \log t)^v}\,dt \\
&\qquad \sim \frac{1}{p}\left\{\sum_{r=n}^{\infty} \frac{x_{rn}}{\log^r p} + \sum_{v=n+1}^{\infty} s_{vn} \sum_{r=v}^{\infty} \frac{x_{rv}}{\log^r p}\right\} \qquad \text{as } p \to 0 \\
&\qquad = \frac{1}{p}\left\{\sum_{r=n}^{\infty} \frac{x_{rn}}{\log^r p} + \sum_{r=n+1}^{\infty} \frac{1}{\log^r p} \sum_{v=n+1}^{r} x_{rv} s_{vn}\right\} \\
&\qquad = \frac{1}{p \log^n p}
\end{aligned}\right\} \qquad \text{.....(A-42)}$$

since, from (A-23)

$$\sum_{v=n+1}^{r} x_{rv} s_{vn} = -x_{rn} \qquad (r > n) \qquad \text{(A-43)}$$

and x_{nn} is unity. Since the asymptotic expansion, which in this case terminates after just one term, is in terms of the gauge functions $\left(\frac{1}{p \log^{n+r} p}\right)$ $(r = 0 \to \infty)$, it follows that the difference between the left and right hand sides of (A-42) is transcendentally small, as $p \to 0$, compared with these gauge functions. That is this difference is $o\left(\frac{1}{p \log^{n+r} p}\right)$ for any r however large; or, putting it another way, it is a function which has a zero asymptotic expansion in terms of these gauge functions. It follows, therefore, with similar significance, that, using (A-26) and (A-28), for n positive (*cf* equation (B-12)), we have the following asymptotic expansion, which also always terminates after one term.

$$\mathcal{L}\left\{\mu_n(\chi,0)\delta(t) + \frac{d\mu_n(\chi,t)}{dt}H(t)\right\} \sim \frac{1}{\log^n p} \qquad \text{as } p \to 0 \qquad \text{(A-44)}$$

where
$$\mu_n(\chi,t) = (-)^n \sum_{v=0}^{\infty} (-)^v \binom{n+v-1}{v} \frac{w_v(\chi)}{(\chi + \log t)^{n+v}} . \qquad \text{(A-45)}$$

Examples of terms which are transcendentally small compared with the gauge functions $\log^{-(n+r)} p$, as $p \to 0$, are

$$p^s (s \geqslant 1)$$

$$\frac{p}{(p + a)}$$

which have inverse transforms $\delta^{(s)}(t)$, $\delta(t) - ae^{-at}$ respectively. These latter functions are transcendentally small compared with the gauge functions $(\chi + \log t)^{-r}$, as $t \to \infty$. If, as seems reasonable, we can assume that this is also true for the inverse transforms of any functions which are transcendentally small compared with the $\log^{-(n+r)} p$, as $p \to 0$, then we can write (*cf* equation (A-20))

$$\theta_{-n}(t) \sim \mu_n(\chi,0)\delta(t) + \frac{d\mu_n(\chi,t)}{dt}H(t) \qquad \text{as } t \to \infty \qquad \text{(A-46)}$$

or equivalently

$$\theta_{-n}(t) \sim \frac{d\mu_n}{dt} \qquad \text{as } t \to \infty . \qquad \text{(A-47)}$$

Thus, substituting from (A-45), we have the following asymptotic expansion which can be used to evaluate* the inverse transform of $\log^{-n} p$, when n is positive.

$$\theta_{-n}(t) \sim (-)^{n+1} \frac{n}{t} \sum_{v=0}^{n} (-)^v \binom{n+v}{v} \frac{w_v(\chi)}{(\chi + \log t)^{n+v+1}} \qquad \text{as } t \to \infty . \qquad \text{(A-48)}$$

---

* With an asymptotic expansion, if the series is divergent, a point will be reached where the addition of further terms increases the error. One must therefore stop at this point.

It should be noted that there will be a limit on the arguments for which this equation is of use. The error might contain some decaying exponential terms and so, even though the series on the right of (A-48) will converge for sufficiently small t, the value it gives for θ_{-n} may be appreciably in error. However there is no reason why the expansion should not be used for negative values of $(X + \log t)$ and indeed it may prove more convenient to take a value of X so that this is so.

As an illustration of this let us consider the associated function

$$\nu_n(t) = -\int_t^\infty \theta_{-n}(\tau)d\tau \ . \tag{A-49}$$

Then

$$\nu_n(t) \sim \mu_n(X,t) \ . \tag{A-50}$$

Taking, for example, a positive value of X and trying to use this asymptotic expansion to evaluate $\nu_1(1)$ it is found to be of no practical use. Thus with $X = 3\gamma$ successive sums are:

Number of terms	1	2	3	4	5	6	7	8
Sum	-0.58	-0.96	-0.90	-0.17	1.17	2.33	1.30	7.65.

and for larger X things get worse (*cf* the behaviour of the $w_r(X)$ as shown in Table 3). However if we take X negative there is no difficulty. Thus with $X = -3\gamma$ the successive sums for $\nu_1(1)$ are:

Number of terms	1	2	3	4	5	6	7	8		20	21
Sum	0.58	1.35	2.06	2.43	2.40	2.23	2.19	2.28	....	2.29	2.22 .

Evidence that the sign of X does not make any change, such as the addition of a constant, to the set of functions for which $\mu_n(X,t)$ is an asymptotic expansion, is given, for example, by consideration of the evaluation of $\nu_1(t)$ for t large. Let us take

$$X = m \log t \tag{A-51}$$

where m is any number, positive or negative, other than -1 such that $|X|$ will also be large. Then, when that is so (*cf* equation (A-34)) we have

$$w_v(X) \approx (-X)^v . \qquad \text{(A-52)}$$

It is then easily seen that the sum, of N terms of the series for $\mu_1(X,t)$, equation (A-45), and hence of the asymptotic expansion for $\nu_1(t)$, is

$$\nu_1(t) \sim \mu_1(X,t) \approx -\frac{1}{\log t}\left\{1 - \left(\frac{m}{m+1}\right)^N\right\} . \qquad \text{(A-53)}$$

Thus we ultimately get the same value whether m is positive or negative (or indeed zero, as (A-45) and Table 3 show) provided $\left|\frac{m}{(m+1)}\right| < 1$.

We have thus established the validity of our expression (A-48) for the inverse transform of $\log^n p$ when n is a negative integer. It should perhaps be noted that we have there an infinity of asymptotic expansion, for the parameter X is arbitrary. For n positive we had previously obtained the expression (A-37) - again containing an arbitrary parameter X . To complete the picture we have the well known fact that

$$\theta_0(t) = \delta(t) \qquad \text{(A-54)}$$

it being the function whose Laplace transform is 1.

A particular case which we require in the main part of the Memorandum is that when $n = 1$ which from (A-37) is

$$\theta_1(t) = -(\gamma + \log t)\delta(t) - \frac{H(t)}{t} . \qquad \text{(A-55)}$$

Appendix B

THE RIGHT HAND DIRAC DELTA FUNCTION AND ITS USE IN LAPLACE TRANSFORM THEORY

Since the Laplace transform integral has a lower limit of zero one has to use what is called a right hand Dirac delta function $\delta(t)$ which is defined to be such that

$$\left.\begin{aligned} \delta^{(n)}(0) &= 0 \\ \delta^{(n)}(\infty) &= 0 \end{aligned}\right\} \qquad n = 0 \to \infty \qquad \text{(B-1)}$$

$$\int_0^t \delta(\tau)f(\tau)d\tau = f(0)H(t) \qquad \text{(B-2)}$$

where

$$\left.\begin{aligned} H(t) &= 0 \qquad t \leqslant 0 \\ &= 1 \qquad t > 0 \end{aligned}\right\} \qquad \text{(B-3)}$$

and $f(t)$ is a fairly good function[15].

It follows that we can write

$$\delta(t) = \frac{dH(t)}{dt} \qquad \text{(B-4)}$$

for

$$\begin{aligned} \int_0^t f(\tau)\frac{dH(\tau)}{d\tau}d\tau &= \Big[f(\tau)H(\tau)\Big]_0^t - \int_0^t \frac{df(\tau)}{d\tau}H(\tau)d\tau \\ &= H(t)f(t) - \int_{0+}^t \frac{df(\tau)}{d\tau}d\tau \\ &= H(t)f(0) + \{H(t) - 1\}\{f(t) - f(0)\} \\ &= H(t)f(0). \end{aligned} \qquad \text{(B-5)}$$

Differentiating (B-2) we see also that $\delta(t)$ is such that

$$\delta(t)f(t) = \delta(t)f(0). \tag{B-6}$$

For the derivatives of δ we have, integrating by parts and using (B-1),

$$\int_0^t \delta^{(n)}(\tau)f(\tau)d\tau = \sum_{r=0}^{n-1} (-)^r f^{(r)}(t)\delta^{(n-1-r)}(t) + (-)^n f^{(n)}(0)H(t) \ . \tag{B-7}$$

Also, from (B-6), by successive differentiation*

$$\delta^{(n)}(t)f(t) = \sum_{r=0}^{n} \binom{n}{r} (-)^r \delta^{(n-r)}(t)f^{(r)}(0) \tag{B-8}$$

and so (B-7) can be rewritten as**

$$\int_0^t \delta^{(n)}(\tau)f(\tau)d\tau = (-)^{n-1} \sum_{r=0}^{n-1} \binom{n}{r+1}(-)^r \delta^{(r)}(t)f^{(n-1-r)}(0) + (-)^n f^{(n)}(0)H(t) \ .$$

$$\ldots\ldots(B\text{-}9)$$

It follows from (B-2) and (B-9), putting $f(\tau) - e^{-p\tau}$, $t = \infty$, and using (B-1), that the Laplace transforms of the δ functions are

$$\mathscr{L}\{\delta^{(n)}(t)\} \equiv \int_0^\infty e^{-pt}\delta^{(n)}(t)dt = p^n \qquad n = 0 \rightarrow \infty \ . \tag{B-10}$$

Consequently, from the well known theorem for the inverse transform of a product, we find that, using (B-9) and (B-7), for $n \geqslant 1$

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* Incidentally putting $t = 0$ in (B-8), we notice that this equation would not be satisfied if the first of the conditions (B-1) in the definition of δ were relaxed.

** We have used the fact that $\displaystyle\sum_{s=0}^{n-1-r} \binom{r+s}{s} = \binom{n}{r+1}$.

$$
\left.
\begin{aligned}
\mathscr{L}^{-1}\left\{p^n \mathscr{L}\left(f(t)\right)\right\} &= \int_0^t f(t-\tau)\delta^{(n)}(\tau)d\tau \\
&= \sum_{r=0}^{n-1} \binom{n}{r+1} \delta^{(r)} f^{(n-1-r)}(t) + f^n(t)H(t) \\
&= \sum_{r=0}^{n-1} f^{(r)}(0)\delta^{(n-1-r)}(t) + f^{(n)}(t)H(t) \quad .
\end{aligned}
\right\} \qquad \text{(B-11)}
$$

The particular case $n = 1$ is the well known formula

$$\mathscr{L}^{-1}\left\{p\mathscr{L}\left(f(t)\right)\right\} = f^{(1)}(t)H(t) + f(0)\delta(t) \qquad \text{(B-12)}$$

which can be written, using (B-10), as

$$p\mathscr{L}\left(f(t)\right) - f(0) = \mathscr{L}\left\{f^{(1)}(t)H(t)\right\}. \qquad \text{(B-13)}$$

In practice one may well obtain as an inverse transform a function in the form of an infinite series of terms, containing the δ function and its derivatives. At the two limits $t = 0, \infty$ such a function is by definition zero, but how do we interpret it elsewhere? Consider for example the function

$$X(t) = \sum_{s=0}^{\infty} \frac{(-)^s \delta^{(s)}(t)}{a^{s+1}} \qquad \left(\mathscr{R}(a) > 0\right) \qquad \text{(B-14)}$$

which has the same Laplace transform as e^{-at} for $p > 0$. Lerch's theorem states that if two functions have the same Laplace transform then their difference is a null function*. Thus for this example we require therefore that

$$\int_0^t \left\{e^{-a\tau} - X(\tau)\right\}d\tau = \frac{1}{a}\left\{1 - e^{-at} + X(t) - H(t)\right\} \qquad \text{(B-15)}$$

---

* A null function is one whose $\int_0^t$ is zero for $t \geqslant 0$.

should be zero for all $t \geqslant 0$. Thus we must have

$$\left.\begin{aligned} X(t) &= e^{-at} && t > 0 \\ &= 0 && t = 0 \end{aligned}\right\} . \qquad \text{(B-16)}$$

That is we must understand the series (B-14) to mean the function (B-16). Similarly any function $\sum_{s=0}^{\infty} b_s \delta^{(s)}(t)$ must be understood as identical with any function whose Laplace transform is $\sum_{s=0}^{\infty} b_s p^s$ (for $p \geqslant 0$)* except possibly at $t = 0$. Now we find that, using (B-8)

$$\left.\begin{aligned} t^n \delta^{(s)}(t) &= (-)^n \frac{s!}{(s-n)!} \delta^{(s-n)}(t) && (0 < n \leqslant s) \\ &= 0 && n > s \end{aligned}\right\} . \qquad \text{(B-17)}$$

and so

$$t^n \sum_{s=0}^{\infty} b_s \delta^{(s)}(t) = (-)^n \sum_{s=0}^{\infty} \frac{(s+n)!}{n!} b_{s+n} \delta^{(s)}(t) \qquad (n = 0 \to \infty) . \qquad \text{(B-18)}$$

Consequently

$$\mathscr{L}t_{t \to \infty}\, t^n \sum_{s=0}^{\infty} b_s \delta^{(s)}(t) = 0 \text{ for any positive or zero } n . \qquad \text{(B-19)}$$

We conclude therefore that any infinite series of δ functions is a representation of a function which is transcendentally small ($t \to \infty$) compared with the

---

* Note the requirement $p \geqslant 0$. If we want the function to be equivalent for all $t > 0$, including $t = \infty$, then their transforms must be identical at $p = 0$ as well as at other values of p . The transform of $\delta^{(s)}(t)$ is p^s at any p including $p = 0$; and from the theorem of equation (5)

$$\mathscr{L}t_{p \to 0}\, \bar{y}(p) = \int_0^{\infty} y(t)dt .$$

If the $\mathscr{R}(a)$ was zero in the above example the transform of e^{-at} at $p = 0$ would not be the same as that of $X(t)$.

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negative powers of t. Thus it can, for example, be a function which is zero for t greater than some finite value, or a function which behaves like $\exp\{-at^b\}$ $(a > 0,\ b > 0)$ as t tends to infinity. The latter possibility is illustrated by the three representative cases $b = \frac{1}{2}, 1, 2$ for which we have the following transforms:

$$\mathscr{L}\left\{\exp(-at^{\frac{1}{2}})\right\} = \frac{1}{p} - \frac{a\sqrt{\pi}}{2p^{3/2}}\exp\left(\frac{a^2}{4p}\right)\left\{1 - \mathrm{erf}\left(\frac{a}{2\sqrt{p}}\right)\right\} \tag{B-20}$$

$$\mathscr{L}\left\{\exp(-at)\right\} = \frac{1}{(p + a)} \tag{B-21}$$

$$\mathscr{L}\left\{\exp(-at^2)\right\} = \frac{\sqrt{\pi}}{2\sqrt{a}}\exp\left(\frac{p^2}{4a}\right)\left\{1 - \mathrm{erf}\left(\frac{p}{2\sqrt{a}}\right)\right\} \tag{B-22}$$

where $\mathrm{erf}(x)$ is the error function*. Of these we cannot identify the first with a power series in p for if we continue it analytically along the negative real axis it becomes complex. The same is true for any value of b less than one and so we can say that $\{\exp -at^b\}$ $(a > 0,\ 1 > b > 0)$ is not equivalent to a series of δ functions. The other two possibilities are quite acceptable; and so finally we can say that any infinite series of δ functions is a representation of a function which either decays, as t tends to infinity, as a simple exponential e^{-at} (or more rapidly), or is zero for t greater than some finite value. The function $X(t)$ of equations (B-14) and (B-16) is an example of the first possibility. An example of the second is the following.

It is well known that the inverse transform of $e^{-p}I_0(p)$ is

$$\frac{1}{\pi\sqrt{t(2 - t)}}\,H(2 - t). \tag{B-23}$$

However, using the power series expansions of e^{-p} and $I_0(p)$ we find that it is

---

* Note that $e^{x^2}\{1 - \mathrm{erf}(x)\} \sim \frac{1}{\sqrt{\pi}x}\left\{1 - \frac{1}{2x^2} + \dots\right\}$ and

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\left\{x - \frac{x^3}{3} + \dots\right\}$$

$$\delta(t) - \delta^{(1)}(t) + \tfrac{3}{4}\,\delta^{(2)}(t) - \tfrac{5}{12}\,\delta^{(3)}(t) + \tfrac{35}{192}\,\delta^{(4)}(t) - \ldots \quad . \tag{B-24}$$

The difference between these two functions must be a null function and so integrating both as before between 0 and t we find that

$$H(t) - \delta(t) + \tfrac{3}{4}\,\delta^{(1)}(t) - \tfrac{5}{12}\,\delta^{(2)}(t) + \tfrac{35}{192}\,\delta^{(3)}(t) - \ldots$$

$$= g(t)H(2-t) + g(2)\{1 - H(2-t)\} \tag{B-25}$$

where*

$$g(t) = \frac{1}{\pi}\left\{\sin^{-1}(1-t) - \frac{\pi}{2}\right\} . \tag{B-26}$$

Thus

$$\zeta(t) \equiv -\delta(t) + \tfrac{3}{4}\,\delta^{(1)}(t) - \tfrac{5}{12}\,\delta^{(2)}(t) + \tfrac{35}{192}\,\delta^{(3)}(t) - \ldots$$

$$= g(t)H(2-t) + 1 - H(2-t) - H(t) \quad . \tag{B-27}$$

This infinite series of δ functions $\zeta(t)$ therefore represents the function on the right hand side of (B-27) which is zero for $t = 0$ and for $t > 2$. More generally, since $\delta\left(\frac{t}{k}\right) = k\delta(t)$, we have

$$\zeta\left(\frac{t}{k}\right) \equiv -k\delta(t) + \tfrac{3}{4}\,k^2\delta^{(1)}(t) - \tfrac{5}{12}k^3\delta^{(2)}(t) + \tfrac{35}{192}k^4\delta^{(3)}(t) - \ldots$$

$$= g\left(\frac{t}{k}\right)H(2k-t) + 1 - H(2k-t) - H(t) . \tag{B-28}$$

It may well be that, for some given series of δ functions, a combination of a finite number of series such as (B-14) and (B-28) can be found which only differs from the given series in a finite number of terms. In such a case there is no point in finding the representation of this remainder finite series as a linear combination of an infinite basic set of infinite series.

* The value of $\sin^{-1}(1-t)$ is that in the range $\frac{\pi}{2} \leqslant \sin^{-1}(1-t) \leqslant \frac{3\pi}{2}$.

Table 1

APPROXIMATE VALUES OF $\Xi(p) = K_1\left(\frac{p}{2}\right) \Big/ \left\{K_0\left(\frac{p}{2}\right) + K_1\left(\frac{p}{2}\right)\right\}$

	Real negative p	Real positive p	Purely imaginary p
$\frac{x}{2}$	$\Xi(-x)$	$\Xi(x)$	$\Xi(\pm ix)$
0	1	1	1
0.1	1.1241 ∓ 0.4753i	0.8024	0.8319 ∓ 0.1723i
0.2	0.7677 ∓ 0.7371i	0.7315	0.7276 ∓ 0.1886i
0.3	0.4605 ∓ 0.6368i	0.6901	0.6650 ∓ 0.1793i
0.4	0.3240 ∓ 0.4781i	0.6622	0.6250 ∓ 0.1650i
0.5	0.2768 ∓ 0.3536i	0.6418	0.5979 ∓ 0.1507i
0.6	0.2672 ∓ 0.2647i	0.6263	0.5788 ∓ 0.1378i
0.7	0.2731 ∓ 0.2012i	0.6139	0.5648 ∓ 0.1264i
0.8	0.2798 ∓ 0.1550i	0.6039	0.5541 ∓ 0.1165i
0.9	0.2959 ∓ 0.1207i	0.5955	0.5459 ∓ 0.1078i
1.0	0.3141 ∓ 0.09482i	0.5884	0.5394 ∓ 0.1003i
2	0.4110 ∓ 0.01063i	0.5512	0.5130 ∓ 0.05769i
3	0.4475 ∓ 0.001360i	0.5362	0.5063 ∓ 0.04000i
4	0.4634 ∓ 0.0001794i	0.5280	0.5037 ∓ 0.03050i
5	0.4718 ∓ 0.00002393i	0.5229	0.5024 ∓ 0.02460i
10	0.4868 ∓ 0.0.......i	0.5199	0.5006 ∓ 0.01245i
∞	0.5	0.5	0.5

Table 2

VALUES OF ρ_r AND $\Gamma^{(r)}(1)/\Gamma(r+1)$

(see Appendix A)

r	ρ_r	$\Gamma^{(r)}(1)/\Gamma(r+1)$
1	-0.577215...	-0.577215...
2	1.644934...	0.989056...
3	-1.202057...	-0.907479...
4	1.082323...	0.981728...
5	-1.036928...	-0.981995...
6	1.017343...	0.993149...
7	-1.008349...	-0.996002...
8	1.004077...	0.998106...
9	-1.002008...	-0.999025...
10	1.000995...	0.999516...
11	-1.000494...	-0.999757...
12	1.000246...	0.999878...
13	-1.000123...	-0.999939...
14	1.000061...	0.999970...
15	-1.000031...	-0.999985...
16	1.000015...	0.999992...
17	-1.000008...	-0.999996...
18	1.000004...	0.999998...
19	-1.000002...	-0.999999...
20	1.000001...	1.000000...

$$\begin{cases} \rho_r = \dfrac{\Psi_r(1)}{\Gamma(r)} = (-)^r \displaystyle\sum_{s=1}^{\infty} \frac{1}{s^r} \qquad (r \geqslant 2) \\ \rho_1 = -\gamma \quad \text{where } \gamma \text{ is Euler's constant} \end{cases}$$

Table 3 - VALUES OF $w_r(X)$

(see Appendix A)

r \ X/Y	-3.0	-2.5	-2.0	-1.5	-1.0	-0.5	0	0.5	1.0	1.5	2.0	2.5	3.0
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2.30886	2.02026	1.73165	1.44304	1.15443	0.865823	0.577216	0.288608	0.000000	-0.288608	-0.577216	-0.865823	-1.15443
2	3.68591	2.43650	1.35367	0.437428	-0.312222	-0.895284	-1.31176	-1.56164	-1.64493	-1.56164	-1.31176	-0.895284	-0.312222
3	3.31853	0.680084	-0.948703	-1.71207	-1.75425	-1.21949	-0.252016	1.00393	2.40411	3.80430	5.06024	6.02772	6.56248
4	-0.368862	-2.57285	-2.32761	-0.715451	1.34781	3.11286	3.99693	3.58372	1.62349	-1.96705	-7.10461	-13.5394	-20.8553
5	-4.60675	-2.11830	1.64326	3.94453	3.49380	0.200765	-5.06373	-10.7080	-14.6598	-14.6066	-8.23596	6.52448	31.2664
6	-1.96868	4.30864	4.66511	-0.511818	-7.38434	-10.9527	-6.92782	6.81290	29.1849	55.2724	76.1208	78.9440	47.7493
7	7.34205	4.23161	-5.94279	-10.8152	-2.70797	16.7792	36.3835	38.1552	2.95636	-82.3604	-216.931	-377.862	-513.047
8	5.06521	-10.7710	-8.92976	12.4549	30.7716	15.9432	-46.9796	-138.377	-194.565	-113.098	224.227	909.673	1950.17
9	-15.5002	-6.45983	24.3226	22.2473	-37.9572	-108.463	-78.1069	161.571	611.450	1053.66	976.771	-415.008	-4060.91
10	-9.48787	32.4055	5.49513	-75.3224	-64.4348	155.940	464.669	401.992	-677.313	-3129.75	-6269.44	-7508.14	-1700.72
11	40.4870	-3.04768	-87.5132	24.5245	292.957	203.048	-803.719	-2362.24	-2271.37	3427.59	18395.7	41315 7	58706.7
12	8.80415	-96.3064	85.3760	271.014	-286.542	-1349.01	-598.988	4930.12	13936.1	14189.9	20685.3	-122871	-301276
13	-115.506	97.3586	214.663	-575.447	-847.131	2306.32	7055.38	657.580	-34757.7	-93616.5	-97648.9	146611	922860
14	45.3137	225.145	-641.043	-166.301	3413.76	1859.65	-17926.8	-40504.5	16761.7	275218	704940	734800	-119913×10
15	315.663	-594.442	105.911	3040.62	-3503.70	-19538.3	7997.86	143293	246307	-298138	-240914×10	-587426×10	-599497×10
16	-396.343	-101.459	2548.94	-5419.34	-10549.6	44768.3	104656	-204710	-119573	-150624×10	401430×10	230469×10^2	535974×10^2
17	-674.863	2389.56	-4856.57	-3809.76	47522.6	-4399.86	-420164	-433400	278814×10	104111×10^2	823925×10	-514738×10^2	-238843×10^3
18	1997.90	-2925.05	-2807.84	36910.0	-63060.2	-293379	668040	352485×10	-448424	-338909×10^2	-937476×10^2	-218254×10^2	666052×10^3
19	244.152	-5389.24	28103.4	-64942.7	-120833	964143	946674	-103944×10^2	-263713×10^2	492179×10^2	402441×10^3	857874×10^3	-464052×10^3
20	-7468.00	20576.9	-41817.3	-49397.4	742683	-102109×10	-899397×10	104042×10^2	125854×10^3	147641×10^3	-992049×10^3	-479783×10^4	-771359×10^4

$w_r(X) \to (-X)^r \quad |X| \to \infty$

γ (Euler's constant) = 0.577215....

LIST OF SYMBOLS

A	inertia matrix (structural + aerodynamic)
A_1	aerodynamic inertia matrix
A_0	see equation (2)
$B\left(\frac{\omega}{v}\right)$	aerodynamic damping matrix
B_∞	$= B(\infty)$
$C\left(\frac{\omega}{v}\right)$	aerodynamic stiffness matrix
C_0	$= C(0)$
C_∞	$= C(\infty)$
D	structural damping matrix $= \frac{d}{dt}$
E	structural stiffness matrix
$G(v\tau)$	constituent of indicial aerodynamic matrix - see equation (2)
$\bar{G}(p)$	Laplace transform of $G(t)$
$\hat{G}(v\tau)$	the difference between $G(v\tau)$ and its asymptotic ($\tau \to \infty$) power series expansion
$H(t)$	right hand Heaviside step function - see equation (B-3)
I	unit matrix
$I_0(z)$	modified Bessel function of the first kind of order zero
$K(\tau)$	indicial aerodynamic matrix - see equations (1) and (2)
$K_0(z)$, $K_1(z)$	modified Bessel functions of the second kind
K_σ	see equation (2), $= - C_0$
K_r	coefficient matrices in the Richardson[1] approximation to $G(v\tau)$
L_r	coefficient matrices in the expansion of $\bar{G}(p)$ - see equation (11)
$v^2M\left(\frac{p}{v}\right)$	the characteristic matrix - see equation (33)
N_r	coefficient matrices in the expansion of $\bar{G}(p)$ - see equation (11)
R	see equation (12)
S	see equation (12)
$S(X)$	$= X^{-1} - I$ (Appendix A)

LIST OF SYMBOLS (continued)

$S_{rs}(v)$	coefficient matrices in $\mathscr{L}^{-1}\left\{M^{-1}\left(\frac{p}{v}\right)\right\}$ - see equation (39)
$S^*_{rs}(v)$	see equations (50) and (51)
T	see equation (12)
$T_{rs}(v)$	coefficient matrices in $M^{-1}\left(\frac{p}{v}\right)$ - see equation (35)
$T^*_{rs}(v)$	see equation (50)
W_i	$= \left(\text{adjoint of } M(u) \times \frac{d\|M\|}{du}\right)_{u=\lambda_i/u}$ - (see equation (34))
W^*_i	see equations (48) and (49)
$X(\chi)$	the infinite lower triangular matrix with non-zero elements given by equation (A-22)
$X_0(\tau)$	$=\mathscr{L}^{-1}\left\{M^{-1}\left(\frac{p}{v}\right)\right\}$, the fundamental solution
$X_j(\tau)$	$= \frac{d^j X_0}{d\tau^j}$
f_r	real constant column matrices which are the coefficients of $\delta^{(r)}(\tau)$ in an instantaneous excitation
p	Laplace transform variable
p_0	constant in Richardson[1] approximation to $G(v\tau)$
$q(\tau)$	column matrix of generalised coordinate
$\tilde{q}$	semi-amplitude of q if motion is simple harmonic of infinite duration
$s_{rs}(\chi)$	elements of the lower triangular matrix S - see Appendix A
t	time
v	airspeed divided by a reference speed
$w_r(\chi)$	elements in the first column of I + S, see equations (A-26) to (A-28)
$x_{ij}(\chi)$	elements of $X(\chi)$
$\Gamma(z)$	the Gamma Function
$\Xi(p)$	$= K_1\left(\frac{p}{2}\right)\Big/\left\{K_0\left(\frac{p}{2}\right) + K_1\left(\frac{p}{2}\right)\right\}$, the Theodorsen function

LIST OF SYMBOLS (continued)

α_i'	left hand characteristic vector
$\alpha_u^{(rs)}$	see equation (43)
β_i	right hand characteristic vector
γ	$= 0.577215\ldots$, Euler's constant
$\delta(t)$	right hand Dirac delta function, see Appendix B
$\delta^{(n)}(t)$	$\dfrac{d^n\delta}{dt^n}$
$\zeta(t)$	see equation (B-23)
ζ_r	see equation (29)
η_r	see equation (29)
$\theta_m(t)$	$= \mathscr{L}^{-1}(\log^m p)$
λ_i	characteristic value
μ_n	see equations (A-44) and (A-45)
ν	$= \omega/v$, frequency parameter
$\xi_n(\chi)$	$= \left[(D + \chi)^n\{\Gamma(z)\}\right]_{z=1}$
ρ_1	$= -\gamma$
ρ_r	$= (-)^r \sum_{s=1}^{\infty} \dfrac{1}{s^r} \quad (r \geqslant 2)$
σ_1	$= \chi - \gamma$
σ_r	$= \rho_r \quad (r \geqslant 2)$
τ	time multiplied by (reference speed/reference length)
χ	arbitrary constant used in getting inverse Laplace transform of $\log^m p$, see Appendix A
$\chi(t)$	see equation (B-14)
$\psi_r(z)$	$= \dfrac{d^r}{dz^r}\{\log \Gamma(z)\}$, the Polygamma functions
ω	frequency multiplied by (reference length/reference speed)
$\mathscr{L}\{f(t)\}$	= Laplace transform of $f(t)$
$\mathscr{L}^{-1}\{\bar{f}(p)\}$	= inverse Laplace transform of $\bar{f}(p)$
$\bar{f}(p)$	$= \mathscr{L}\{f(t)\}$

LIST OF SYMBOLS (concluded)

$\binom{n}{r}$	$= n!/r!(n-r)!$
$f^{(n)}(x)$	$= \frac{d^n f}{dx}$
$\mathscr{I}(x)$	signifies the imaginary part of x
$\mathscr{R}(x)$	signifies the real part of x

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17. Abstract

The solution of the subsonic flutter problem, when the commonly used linear differential equation model is replaced by the more correct linear integro-differential equation model, is studied and the nature of the system's free motion established. The different forms appropriate to two-dimensional and three-dimensional flow, and to the cases when the system has a zero characteristic value, are developed in detail. It is shown that, for large time t, the behaviour can variously be like 1, $\log^{-1} t$, t^{-2} or t^{-3}.